

# ON UNCONDITIONAL BASISNESS OF B-QUASI-EXPONENTIALS

ARKADI MINKIN

ABSTRACT. We consider operator  $A = K^{-1}$ ,  $Kh = Bh + (h, f)g$  in separable Hilbert space  $\mathfrak{H}$  where  $B$  is quasinilpotent operator whose resolvent is entire operator function of exponential type  $a$ . Let  $\varphi(z)$  be Fredholm determinant of  $K$ . We prove two necessary conditions for unconditional basisness of eigenfunctions of  $A$  without any a priori restrictions on its spectrum  $\Lambda$ : weights  $w^2(\lambda)$  and  $W^2(\lambda)$  on  $\mathbb{R}$  must satisfy Muckenhoupt condition  $(A_2)$ .

Here  $w^2(\lambda) := \|g(\lambda)\|^2$  and  $W^2(\lambda) := |\varphi(\lambda)|^2 / (\delta(\lambda) \cdot w^2(\lambda))$  where vector-valued function  $g(z) := (I - zB)^{-1}g$  and  $\delta(z)$  is regularized distance from point  $z \in \mathbb{C}$  to  $\Lambda$ .

## INTRODUCTION

One of the main challenges of spectral theory is question of similarity to normal for non-selfadjoint operators. If operator  $A$  has compact resolvent the problem reduces to *unconditional basisness* of its eigenfunctions and we will write  $A \in (\text{UB})$ . A celebrated example is question of unconditional basisness of exponentials or more generally of values of reproducing kernels deeply investigated in the classical papers of B.S.Pavlov, S.V.Hruščev and N.K.Nikol'skii [25], [10],[21], giving birth to *projection* method, see also their excellent survey article [12].

Similarity to normal problem has been attacked from various directions, for dissipative as well as for non-dissipative operators, see for example works of V.E.Katsnel'son, B.S.Pavlov, N.K.Nikol'skii, V.I.Vasyunin, S.Treil, S.Kupin [13], [24],[23],[22],[14] to name a few. We refer the reader to [20] and [23] for more details.

Recently G. M. Gubreev substantially developed *projection* method in a series of remarkable publications [3]–[6] and others. He introduced classes  $\Sigma^{(exp)}$  and  $\Lambda^{(exp)}$  of quasinilpotent operators with exponential resolvent's growth and considered their one- and finite-dimensional perturbations continuing A.P.Khromov's investigations of perturbations of Volterra operators [15].

However G.M.Gubreev imposed several a priori restrictions on the operators in question. Our goal is to deduce basic necessary conditions

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for unconditional basisness of eigenfunctions *without any a priori hypotheses*. In order to describe existing results in more details we will introduce some notations sticking to those from [4] and [5].

Let  $\Sigma^{(exp)}$  denote the set of all bounded linear operators  $B$  in a separable Hilbert space  $\mathfrak{H}$ , satisfying the following conditions:

- (i) spectrum  $\sigma(B) = \{0\}$ ,  $\ker B = \{0\}$ ;
- (ii)  $\ker B^* = \{0\}$ ;
- (iii)  $(I - zB)^{-1}$  is of finite exponential type  $a > 0$ ;
- (iv) the semigroup of operators  $V(t) := \exp(-iB^{-1}t)$ ,  $t \geq 0$  is of class  $C_0$ .

Recall that the latter means that  $V(t)$  is continuous in strong operator topology for  $t \geq 0$  and  $\lim_{t \rightarrow +0} V(t)h = h$ ,  $\forall h \in \mathfrak{H}$ . Obviously such  $B$  is quasinilpotent due to condition (iii).

Let  $\Lambda^{(exp)}$  be the set of all bounded linear operators  $B$  in  $\mathfrak{H}$ , satisfying conditions (i), (iii) above, whereas conditions (ii) and (iv) are replaced by dissipativity of  $B$

$$(ii') \quad (1/2i)(B - B^*) \geq 0.$$

Note that for dissipative  $B$  the subspace  $\ker B$  reduces it whence automatically  $\ker B^* = \ker B = \{0\}$  for  $B \in \Lambda^{(exp)}$  and (ii) is valid too. So  $\Lambda^{(exp)} \subset \Sigma^{(exp)}$ . Moreover for any automorphism  $S$  of  $\mathfrak{H}$  we have

$$S\Lambda^{(exp)}S^{-1} := \{SBS^{-1}, B \in \Lambda^{(exp)}\} \subset \Sigma^{(exp)}.$$

**Definition 0.1.** A vector-valued function of the form

$$g(z) = g(z, B) := (I - zB)^{-1}g, \quad B \in \Sigma^{(exp)}, \quad g \in \mathfrak{H},$$

is called a  $B$ -quasi-exponential or merely quasi-exponential if the operator is known from the context.

**Definition 0.2** (Gubreev).  $B$ -quasi-exponential  $g(\cdot)$  is said to be right-regular if there exists constant  $M > 0$  such that

$$\int_{\mathbb{R}} |(g(x), h)|^2 \|g(x)\|^{-2} dx \leq M \|h\|^2, \quad \forall h \in \mathfrak{H}$$

and is said to be regular if there exist constants  $m, M > 0$  such that

$$m \|h\|^2 \leq \int_{\mathbb{R}} |(g(x), h)|^2 \cdot \|g(x)\|^{-2} dx \leq M \|h\|^2, \quad \forall h \in \mathfrak{H}.$$

**Definition 0.3** (Gubreev). A pair of vectors  $f, g \in \mathfrak{H}$  is said to be compatible with an operator  $B$  of class  $\Sigma^{(exp)}$  if  $\ker K = \ker K^* = \{0\}$  for operator

$$(0.1) \quad Kh := Bh + (h, f)g, \quad h \in \mathfrak{H}.$$

In this case we will consider a closed densely defined operator  $A := K^{-1}$ . Its action may be also described as follows:

$$(0.2) \quad \text{if } h = l + (B^{-1}l, f)g \quad \text{then } Ah = B^{-1}l, \quad l \in \mathfrak{L} := B\mathfrak{H}.$$

The set of such operators  $A$  will be denoted  $\mathcal{K}_\Sigma$ . In case we take  $B$  from  $\Lambda^{(exp)}$  we will denote  $\mathcal{K}_\Lambda$  the set of respective operators  $A$ .

For  $B \in \Sigma^{(exp)}$  introduce another quasi-exponential

$$f_*(z) := f(z, B_*) = (I - zB_*)^{-1}f, \quad B_* := -B^*.$$

**Remark 0.4.** Note that  $B_* \in \Sigma^{(exp)}$  because the semigroup

$$V(t)^* \equiv \exp(-iB_*^{-1}t)$$

is also of class  $C_0$ , see Corollary 1.3.2 in [19]. Therefore  $A \in \mathcal{K}_\Sigma$  implies  $A_* := -A^* \in \mathcal{K}_\Sigma$ .

To avoid unnecessary complications throughout the paper we will always assume *simplicity of eigenvalues* of  $A$ . Set  $z_* := -\bar{z}$ . Then the resolvent of  $A \in \mathcal{K}_\Sigma$  takes the form

$$(0.3) \quad \begin{aligned} (A - zI)^{-1}h &= B(I - zB)^{-1}h + \mathcal{L}(z)h \\ \mathcal{L}(z)h &:= \varphi^{-1}(z) \cdot (h, f_*(z_*))g(z) \\ \varphi(z) &:= 1 - z(g(z), f). \end{aligned}$$

Denote  $\Lambda = \{\lambda_k\}$  the spectrum of  $A$  enumerated in the order of non-decreasing absolute values. Of course  $\Lambda = \varphi^{-1}(0)$ . A. P. Khromov pointed out to us that  $\varphi(z)$  is Fredholm determinant [26, §6.5.2] of  $K = A^{-1}$ . Note also that eigenfunctions of  $A$  and  $A^*$  coincide respectively with vectors

$$(0.4) \quad \{g(\lambda_k)\}_{\lambda_k \in \Lambda}$$

and

$$(0.5) \quad \{f_*(\lambda_{k*})\}_{\lambda_k \in \Lambda}.$$

In the sequel we will frequently use the linear resolvent growth condition:

$$(LRG) \quad \|R_z(A)\| \asymp d(z), \quad d(z) := \text{dist}(z, \Lambda)$$

which stems from  $A \in (\text{UB})$ . Indeed, similarity of  $A$  to normal operator  $N$  means that  $R_z(A) = SR_z(N)S^{-1}$  for some bounded and boundedly invertible operator  $S$ . Then  $\|R_z(A)\| \leq \|S\| \cdot \|S^{-1}\| \cdot \|R_z(N)\|$ . Interchanging  $A$  and  $N$  we get inverse inequality and thus arrive to (LRG) because  $\|R_z(N)\| = d(z)$ .

Note also a helpful formula

$$(0.6) \quad -iB(I - zB)^{-1}h = \int_0^a e^{izt}V(t)h dt, \quad h \in \mathfrak{H}, \quad z \in \mathbb{C}.$$

whence for any fixed  $c > 0$

$$(0.7) \quad \|R_z(B)\| \leq M/(1 + |\text{Im } z|), \quad \text{Im } z \geq -c, \quad M = M(c).$$

Therefore  $(I - zB)^{-1}$  has exponential type  $a > 0$  in the lower half-plane and is bounded in the upper half-plane. In addition mention a helpful formula

$$(0.8) \quad \int_{\mathbb{R}} \|R_\lambda(B)h\|^2 d\lambda \leq \frac{1}{2\pi} \int_0^a \|V(t)h\|^2 dt \leq K \|h\|^2,$$

where  $K = \sup_{0 \leq t \leq a} \|V(t)\|^2 / (2\pi)$ .

## 1. GUBREEV'S RESULTS.

G.M.Gubreev invented an integral transformation related to each Muckenhoupt weight  $w^2(\lambda)$  on  $\mathbb{R}$ . Recall that this condition for a positive weight  $v(\lambda)$ ,  $\lambda \in \mathbb{R}$  reads as follows, see Lemma 4.2 in [12, Part III]:

$$(A_2) \quad \sup_{z \in \mathbb{C}_+} v(z) \cdot (v^{-1})(z) < \infty.$$

Here  $\mathbb{C}_+$  is the upper half-plane and  $v(z)$  stands for harmonic continuation of  $v(\lambda)$  to  $z \in \mathbb{C}_+$ . For a Muckenhoupt weight  $w^2$  G.M.Gubreev [4, §2] defined function  $y_w(t)$  on  $\mathbb{R}_+$  and introduced transformation

$$(\mathcal{D}_w f)(u) := \frac{1}{\sqrt{2\pi}} \int_0^\infty y_w(u, t) f(t) dt.$$

It maps  $L^2(\mathbb{R}_+)$  isomorphically onto weighted Hardy space  $H_+^2(w^{-2})$  [4] and is bounded from below. Notice also that Gubreev's transformation  $\mathcal{D}_w$  maps  $L^2(0, a)$  isomorphically onto Hilbert space  $A_a^2(\mathbb{R}, w^{-2})$  [4] of entire functions  $F(z)$  of exponential type such that

- $h_F(\pi/2) \leq 0$ ,  $h_F(-\pi/2) \leq a$ ;
- $(F, G)_{A_a^2(\mathbb{R}, w^{-2})} := \int_{\mathbb{R}} F(\lambda) \cdot \overline{G(\lambda)} \cdot w^{-2}(\lambda) d\lambda$ ,  $F, G \in A_a^2(\mathbb{R}, w^{-2})$ .

Recall that for entire function  $F(z)$  of exponential type its indicator function is

$$h_F(\theta) := \limsup_{r \rightarrow \infty} r^{-1} \log |F(re^{i\theta})|, \quad -\pi < \theta \leq \pi.$$

Denote  $J_a$  the integration operator in  $L^2(0, a)$ :

$$(J_a f)(x) := i \int_0^x f(s) ds, \quad f \in L^2(0, a)$$

and introduce  $w$ -quasi-exponential

$$y_w^a(z, t) := ((I - zJ_a)^{-1} y_w)(t), \quad 0 \leq t \leq a; \quad z \in \mathbb{C}.$$

With an operator  $A \in K_\Sigma$  we associate two weights on the real line:

$$(1.1) \quad w^2(x) := \|g(x)\|^2 = \|(I - xB)^{-1} g\|^2,$$

$$(1.2) \quad w_*^2(x) := \|f_*(x_*)\|^2 = \|(I + xB^*)^{-1} f\|^2.$$

Let us agree that throughout the paper all operations on sets are understood in element-wise way. Now we are in position to formulate Gubreev's results.

**Theorem A** (Gubreev, theorem 0.2 in [5]). *Let  $A \in \mathcal{K}_\Sigma$  and assume that*

$$(1.3) \quad \inf \operatorname{Im} \Lambda > 0.$$

*Then  $A \in (\text{UB})$  iff the following conditions are satisfied:*

$$(G1) \quad w^2(x) \in (A_2);$$

$$(G2) \quad |\varphi(\lambda)|^2 / w(\lambda)^2 \in (A_2);$$

$$(G3) \quad B \text{ is isomorphic to } J_a : B = S J_a S^{-1};$$

$$(G4) \quad S y_w^a = g;$$

$$(G5) \quad h_\varphi(\pi/2) = 0, \quad h_\varphi(-\pi/2) = a;$$

$$(G6) \quad \Lambda \in (C);$$

*where  $S$  is an isomorphism from  $L^2(0, a)$  onto  $\mathfrak{H}$ .*

Recall that for a sequence of distinct complex numbers  $\{\mu_k\}_{k=-\infty}^\infty$  in  $\mathbb{C}_+$  (or in  $\mathbb{C}_-$ ) the Carleson condition (C) means [2, Ch. 7]

$$\inf_k \prod_{j \neq k} \left| \frac{\mu_k - \mu_j}{\mu_k - \bar{\mu}_j} \right| > 0.$$

This theorem remains also valid for the case of multiple spectrum provided that eigenvalues multiplicities are uniformly bounded. Of course condition (1.3) may be replaced by spectrum semi-boundedness  $\inf \operatorname{Im} \Lambda > -\infty$  with appropriate change in formulations but still it is too restrictive.

**Remark 1.1.** *Note that important estimate (4.12) of  $\mathcal{L}(z)$  from below in [4] uses multiplicative representation of  $\varphi(z)$  (same as ours (3.2)) and unnamed formula after it:*

$$\|\mathcal{L}(z)\| \geq \operatorname{Im} \frac{\varphi'(z)}{\varphi(z)} > d$$

*where  $z$  lies in a certain half-plane. However the second inequality above is correct only for semi-bounded spectrum.*

In subsequent article [5] G.M.Gubreev offered important refinement of theorem A by replacing (1.3) with requirement that there is a strip free of spectrum:

$$(G0) \quad \boxed{\operatorname{dist}(\Lambda, \mathbb{R}) > 0.}$$

Let  $\Lambda_\pm := \Lambda \cap \mathbb{C}_\pm$ . We will need a counterpart of (G6):

$$(G6') \quad \Lambda_\pm \in (C).$$

**Theorem B** (Gubreev, theorem 0.3 in [5]). *Let  $A \in \mathcal{K}_\Lambda$ . Assume that  $g(\cdot)$  is right-regular and the spectrum  $\Lambda$  satisfies (G0). Then  $A \in (\text{UB})$  iff conditions (G1)–(G6') are valid where (G6') replaces (G6).*

Thus spectrum restrictions were *nearly removed* at expense of imposing a priori *right-regularity* of  $g(\cdot)$ . This condition seems to be difficult to check unless we consider basisness of  $w$ -quasi-exponentials [3] or *regular* de Branges space  $\mathcal{H}(E)$  [7] where this property is actually built in definitions.

**Remark 1.2.** Recall that these general results of G.M.Gubreev contain many earlier achievements of other authors. Indeed, if we choose  $\mathfrak{H} = L^2(0, a)$ ,  $B = J_a$ , take  $g = y_w^a$  for some Muckenhoupt weight  $w^2$  and require  $\Lambda \subset \mathbb{C}_+$ , then theorem A gives criterion of unconditional basisness of  $w$ -quasi-exponentials, equivalent to theorem 1 in [11] for  $\theta(z) \equiv \exp(iaz)$ .

For  $w(\lambda) \equiv 1$   $w$ -quasi-exponentials become usual exponentials. Thus theorem A reduces to B.S.Pavlov's result [25] whereas theorem B will coincide with our main theorem [18] though the case of unconditional basisness in the span in the latter article remained uncovered.

**Remark 1.3.** Note that full criterion of unconditional basisness of exponentials in  $L^2(0, a)$  without spectrum restriction [17] does not follow from theorem B. Nevertheless, applying his approach to operators in de Branges space  $\mathcal{H}(E)$ , G.M.Gubreev announced important results concerning basisness of their reproducing kernel bases: theorem 8 in [8] (sufficient conditions) and theorem 1 in [9] (criterion without spectrum restrictions).

The latter result supersedes that of [17] and partially extends N.K.Nikolskii's criterion [21] to the case of arbitrary spectrum  $\Lambda \subset \mathbb{C}$ . Indeed, de Branges space  $\mathcal{H}(E)$  is isometrically isomorphic to the special model subspace  $K_\Theta$  in  $H_+^2$ , namely  $\Theta(z) := \overline{E(\bar{z})}/E(z)$  – inner function meromorphic in the whole plane.

**Remark 1.4.** Recall that the main tool in [25] was boundedness of certain skew projector built via generating function of the system of exponentials. This function was successfully applied to the question of basisness of scalar and vector exponentials, see e.g. [1], [16].

But if  $A \in \mathcal{K}_\Sigma$  it must be some operator-valued entire function  $M(z)$  such that  $M(\lambda_k)g(\lambda_k) = 0$ ,  $\lambda_k \in \Lambda$ . However, to the best of our knowledge neither its existence nor criterion of boundedness of Hilbert transform with operator weight were established to date.

## 2. MAIN RESULT

In the present paper we establish necessity of (G1),(G2) (the latter condition is suitably changed) *without* (G0) and *right-regularity* of  $g(\cdot)$  which sets foundation to obtain full criterion for  $A \in (\text{UB})$  without any a priori hypotheses.

Introduce regularized distance

$$(2.1) \quad \delta(z) := d(z)/(1 + d(z)), \quad z \in \mathbb{C}$$

and define weight

$$W(\lambda) := \frac{|\varphi(\lambda)|}{w(\lambda) \cdot \delta(\lambda)}, \quad \lambda \in \mathbb{R}.$$

**Theorem 2.1.** *If  $A \in \mathcal{K}_\Sigma$  then the following statements hold true.*

- (i)  $A \in (\text{UB}) \implies g(\cdot)$  is right-regular  $\implies w^2 \in (A_2)$ ;
- (ii)  $A \in (\text{UB}) \implies f_*(\cdot)$  is right-regular  $\implies w_*^2 \in (A_2)$ ;
- (iii)  $A \in (\text{UB}) \implies W^2 \in (A_2)$ .

Recall that we don't restrict zeros of  $\varphi(z)$  besides that they are simple. However, without loss of generality we assume that there are no real ones.

Throughout the paper we will use the following notations.

- $c, C, \varepsilon$  stand for different positive constants which may vary even during a single computation;
- $x \lesssim y$  means that  $x \leq Cy$ , constant  $C$  doesn't depend on indeterminates  $x, y$  and  $x \asymp y$  is equivalent to  $x \lesssim y$  and  $y \lesssim x$ .

Let us describe an outline of the paper. In section 3, lemma 3.3, we give a double-sided estimate for  $\mathcal{L}(z)$ . Emphasize that its predecessor lemma 3.2 is the key element in the whole proof. It provides a lower bound for  $\mathcal{L}(z)$  which is unattainable by methods from [4],[5] if spectrum  $\Lambda$  is arbitrary. Then in section 4 we establish weighted estimates of the integral over  $\mathbb{R}$  of  $R_\lambda(A)$ ,  $\mathcal{L}(\lambda)h$  and  $\mathcal{L}^*(\lambda)h$ . These estimates are obtained by refining G.M.Gubreev's reasoning, namely we split  $\mathbb{R}$  into several subsets and argue differently for each of them. In the last section 5 our main result – theorem 2.1 is proved.

### 3. ESTIMATE OF FINITE-MEROMORPHIC RESOLVENT'S PART $\mathcal{L}(z)$

A more precise formula for  $\varphi(z)$  reads as follows

$$(3.1) \quad \varphi(z) = 1 - z(g, f) + iz^2 \int_0^a e^{izt} (V(t)g, f) dt,$$

Thus  $\varphi(z)$  belongs to the class Cartwright, assumes multiplicative representation

$$(3.2) \quad \varphi(z) = e^{idz} \prod'_{\lambda_k \in \Lambda} (1 - z/\lambda_k)$$

and

$$h_\varphi(+\frac{\pi}{2}) = -d + \frac{l}{2} \leq 0, \quad h_\varphi(-\frac{\pi}{2}) = d + \frac{l}{2} \leq a,$$

whence  $d \geq l/2$ . Here  $l$  is the *width* of the indicator diagram of  $\varphi(z)$ , i.e length  $|I_\varphi|$  of the interval  $I_\varphi \subset i\mathbb{R}$  [16, §17.2, p.127].

**Lemma 3.1.** *If  $A \in \mathcal{K}_\Sigma$  and has infinite discrete spectrum then  $l > 0$  and therefore  $d > 0$  too.*

*Proof.* Assume on the contrary that  $l = 0$ . Take two eigenvalues  $\lambda_{1,2}$  and define function

$$\psi(z) := \frac{\varphi(z)}{(z - \lambda_1)(z - \lambda_2)}.$$

Obviously  $|I_\psi| = 0$ . From (3.1) it follows that  $\psi(x) \in L^2(\mathbb{R})$ . Then by Paley-Wiener theorem  $\psi(z) \equiv 0$  and so is  $\varphi(z)$ .  $\square$

Following [4, p.908] observe that

$$(3.3) \quad (g(\lambda), f_*(\mu_*)) = (\varphi(\lambda) - \varphi(\mu))/(\mu - \lambda)$$

whence

$$(3.4) \quad (g(\lambda_k), f_*(\mu_*)) = -\varphi(\mu)/(\mu - \lambda_k), \quad \lambda_k \in \Lambda$$

and

$$(3.5) \quad (g(\lambda_k), f_*(\lambda_{j*})) = -\delta_{jk} \cdot \varphi'(\lambda_k).$$

Introduce functions

$$\varphi_k(\lambda) := \frac{\varphi(\lambda)}{(\lambda - \lambda_k)\varphi'(\lambda_k)}.$$

In the formulae (3.6)-(3.9) below we assume that  $A \in (\text{UB})$ . Expand  $f_*(\mu_*)$  over eigenfunctions (0.5) of  $A^*$ :

$$(3.6) \quad f_*(\mu_*) = \sum_j \overline{\varphi_j(\mu)} \cdot f_*(\lambda_{j*}).$$

Then

$$(3.7) \quad \|f_*(\mu_*)\|^2 \asymp \sum_j |\varphi_j(\mu)|^2 \|f_*(\lambda_{j*})\|^2.$$

Quite analogously, expanding  $g(\lambda)$  over (0.4),

$$(3.8) \quad g(\lambda) = \sum_k \varphi_k(\lambda) g(\lambda_k)$$

we get

$$(3.9) \quad \|g(\lambda)\|^2 \asymp \sum_k |\varphi_k(\lambda)|^2 \|g(\lambda_k)\|^2.$$

From  $A \in (\text{UB})$  follows uniform minimality of eigenfunctions, i.e.

$$(UM) \quad \|f_*(\lambda_{k*})\| \|g(\lambda_k)\| \asymp |\varphi'(\lambda_k)|.$$

**Lemma 3.2.** *If  $A \in (\text{UB})$ , then  $\|\mathcal{L}(z)\| \geq C > 0$ ,  $z \in \mathbb{C}$ .*

*Proof.* Observe that

$$-\text{tr } \mathcal{L}(z) = \frac{\varphi'(z)}{\varphi(z)} = \sum' \frac{1}{z - \lambda_k} + \text{id}.$$



Applying Cauchy-Schwarz inequality we get

$$\begin{aligned}
 \left| \sum' \frac{1}{z - \lambda_k} \right| &\equiv \left| \sum' \frac{\varphi_k(z)}{\varphi(z)} \cdot \varphi'(\lambda_k) \right| \\
 &\lesssim \frac{1}{|\varphi(z)|} \left[ \sum |\varphi_k(z)|^2 \|g(\lambda_k)\|^2 \right]^{1/2} \cdot \left[ \sum |\varphi_k(z)|^2 \|f_*(\lambda_{k*})\|^2 \right]^{1/2} \\
 &\asymp \frac{\|g(z)\| \cdot \|f_*(z_*)\|}{|\varphi(z)|} \asymp \|\mathcal{L}(z)\|,
 \end{aligned}
 \tag{3.10}$$

where we used (UM) and norm estimates (3.7) and (3.9). From (3.10) and lemma 3.1 stems

$$0 < d \leq |\operatorname{tr} \mathcal{L}(z)| + C \|\mathcal{L}(z)\| \lesssim \|\mathcal{L}(z)\|,$$

and the lemma is proved.  $\square$

Next let us establish double-sided estimate of  $\mathcal{L}(z)$ .

**Lemma 3.3.** *If  $A \in (\text{UB})$ , then*

$$(3.11) \quad \boxed{\|\mathcal{L}(z)\| = \frac{\|g(z)\| \cdot \|f_*(z_*)\|}{|\varphi(z)|} \asymp \frac{1}{\delta(z)}, \quad |\operatorname{Im} z| \leq c, \quad c > 0.}$$

*Proof.* The upper estimate stems directly from relation (LRG) and estimate (0.7) of the Fredholm resolvent of operator  $B$ .

Let us turn to the estimate from below. Due to lemma 3.2 it is enough to prove it in small circles of radii  $\varepsilon$  around eigenvalues intersecting with the horizontal strip  $|\operatorname{Im} z| \leq c$ . However in these circles the resolvent  $R_z(B)$  is uniformly bounded from above, see (0.7). Together with (LRG) it yields lemma's assertion.  $\square$

**Corollary 3.4.** *If  $A \in (\text{UB})$ , then*

$$(3.12) \quad \frac{1}{\|g(\lambda)\|} \asymp \frac{\delta(\lambda) \cdot \|f_*(\lambda_*)\|}{|\varphi(\lambda)|}, \quad \lambda \in \mathbb{R}.$$

#### 4. WEIGHTED ESTIMATE OF RESOLVENT'S INTEGRAL

**Lemma 4.1.** *If  $A \in (\text{UB})$  then*

$$(4.1) \quad \sup_{\lambda_k \in \Lambda} \int_{\mathbb{R}} \frac{\delta^2(\lambda)}{|\lambda_k - \lambda|^2} d\lambda < \infty.$$

*Proof.* Recall that always  $\delta(\lambda) < 1$  and consider two cases:

- (i)  $|\operatorname{Im} \lambda_k| \geq 1$  ;
- (ii)  $|\operatorname{Im} \lambda_k| < 1$ .

*Case (i).* Remove  $\delta^2(\lambda)$  from (4.1) and conclude that the integral  $\leq \pi$ .  
*Case (ii).* We can assume for deficiency that  $0 < \text{Im } \lambda_k < 1$ . Fix  $k$  and divide  $\mathbb{R}$  into two sets:

$$\Xi_0 = \{|\lambda - \text{Re } \lambda_k| \geq 1\}, \quad \Xi_1 = \mathbb{R} \setminus \Xi_0.$$

Taking an integral in (4.1) over  $\Xi_0$  and replacing there  $\delta(\lambda)$  by 1 we conclude that it is bounded by an absolute constant. Further, take the same integral over  $\Xi_1$  and use trivial bound  $\delta(\lambda) < d(\lambda)$ . Set  $x = \text{Re } \lambda_k$ ,  $y = \text{Im } \lambda_k > 0$ . Then we need to estimate an integral

$$\int_{|\lambda-x| \leq 1} \frac{d(\lambda)^2}{(\lambda-x)^2 + y^2} d\lambda$$

which is  $\leq 2$  because  $d(\lambda) \leq \text{dist}(\lambda, \lambda_k) = \sqrt{(\lambda-x)^2 + y^2}$ .  $\square$

**Lemma 4.2.** *If  $A \in (\text{UB})$ , then*

$$(4.2) \quad \int_{\mathbb{R}} \|(A - \lambda)^{-1}\|^2 \cdot \delta(\lambda)^2 d\lambda \lesssim \|h\|^2.$$

*Proof.* Expand  $h$  into eigenfunctions

$$(4.3) \quad h = \sum_k h_k g(\lambda_k) / \|g(\lambda_k)\|.$$

Then

$$(A - \lambda)^{-1} h = \sum_k h_k \frac{g(\lambda_k)}{\lambda_k - \lambda} \cdot \frac{1}{\|g(\lambda_k)\|}$$

and

$$\|(A - \lambda)^{-1} h\|^2 \asymp \sum_k |h_k|^2 \frac{1}{|\lambda_k - \lambda|^2}.$$

Therefore we have

$$(4.4) \quad \int_{\mathbb{R}} \|(A - \lambda)^{-1} h\|^2 \cdot \delta^2(\lambda) d\lambda \asymp \sum_k |h_k|^2 \int_{\mathbb{R}} \frac{\delta^2(\lambda)}{|\lambda_k - \lambda|^2} d\lambda$$

and it suffices to apply inequality (4.1).  $\square$

**Lemma 4.3.** *If  $A \in (\text{UB})$  then the following estimates hold true*

$$(4.5) \quad \int_{\mathbb{R}} \delta^2(\lambda) \frac{1}{|\varphi(\lambda)|^2} \|g(\lambda)\|^2 \cdot |(h, f_*(\lambda_*))|^2 d\lambda \lesssim \|h\|^2,$$

$$(4.6) \quad \int_{\mathbb{R}} \delta^2(\lambda) \frac{1}{|\varphi(\lambda)|^2} \|f_*(\lambda_*)\|^2 \cdot |(h, g(\lambda))|^2 d\lambda \lesssim \|h\|^2.$$

*Proof.* Let us begin with (4.5). We have

$$\|R_\lambda(A)h\| \geq \|\mathcal{L}(\lambda)h\| - \|R_\lambda(B)h\|$$

whence

$$\|\mathcal{L}(\lambda)\|^2 \leq 2 [\|R_\lambda(A)h\|^2 + \|R_\lambda(B)h\|^2].$$

Thus

$$\int_{\mathbb{R}} \delta^2(\lambda) \cdot \|\mathcal{L}(\lambda)h\|^2 d\lambda \leq C \|h\|^2 + K \|h\|^2,$$

where we took into account (4.2) and (0.8).

The second inequality (4.6) is proved along the same lines.  $\square$

## 5. PROOF OF THE THEOREM 2.1.

Recall that items (i) – (iii) below are those from this theorem.

*First implication in the statement (i).* Combining (3.12) and (4.6), we obtain the desired inequality

$$\int_{\mathbb{R}} \frac{|(g(\lambda), h)|^2}{\|g(\lambda)\|^2} d\lambda \lesssim \int_{\mathbb{R}} \delta^2(\lambda) \frac{1}{|\varphi(\lambda)|^2} \|f_*(\lambda_*)\|^2 \cdot |(g(\lambda), h)|^2 d\lambda \lesssim \|h\|^2.$$

Next, we will need a lemma which is a counterpart of Lemma 4.1 in [4]. Its proof is essentially the same as in [4]. For the reader's sake we reproduce it here with necessary changes, putting the factor  $\delta(\lambda)$  in appropriate places.

**Lemma 5.1.** *If  $A \in (\text{UB})$ , then the following estimates hold true*

$$(5.1) \quad \int_{\mathbb{R}} w^{\pm 2}(x)(1+x^2)^{-1} dx < \infty.$$

*Proof.* Convergence of the first integral stems from the identity [4, p.908]

$$(5.2) \quad \int_{\mathbb{R}} x^{-2} \|g(x) - g(0)\|^2 dx = \int_{\mathbb{R}} \|R_x(B)g\|^2 dx \leq K \|g\|^2.$$

Estimate the second integral. From (3.12) stems that

$$(5.3) \quad \int_{\mathbb{R}} w^{-2}(x)(1+x^2)^{-1} dx \lesssim \int_{\mathbb{R}} \delta^2(x) |\varphi(x)|^{-2} \|f_*(x_*)\|^2 (1+x^2)^{-1} dx.$$

Following [4, p.909-910], we put in (4.6)  $h = f_*(\mu_*)$  for  $\mu = \mu_1 \notin \sigma(A)$  and  $\mu = \mu_2 \in \sigma(A)$ . From (3.3) and (3.4) it follows that

$$\begin{aligned} \int_{\mathbb{R}} \delta^2(\lambda) |\varphi(\lambda)|^{-2} \|f_*(\lambda_*)\|^2 \cdot |\varphi(\lambda) - \varphi(\mu_1)|^2 |\lambda - \mu_1|^{-2} d\lambda &< \infty, \\ \int_{\mathbb{R}} \delta^2(\lambda) \|f_*(\lambda_*)\|^2 \cdot |\lambda - \mu_2|^{-2} d\lambda &< \infty, \end{aligned}$$

whence

$$\int_{\mathbb{R}} \delta^2(\lambda) |\varphi(\lambda)|^{-2} \cdot \|f_*(\lambda_*)\|^2 |\lambda - \mu_1|^{-2} d\lambda < \infty.$$

Plugging this estimate into (5.3) yields the desired result.  $\square$

*Second implication in the statement (i).* Recall that its counterpart is Proposition 4.2 in [4] whose demonstration is based on the following assumptions:

- (a)  $B \in \Sigma^{(exp)}$ ;
- (b)  $g(\cdot)$  is *right-regular*;
- (c) integrals from Lemma 4.1 in [4] converge;
- (d) validity of estimate (4.19) from Lemma 4.3 in [4].

However (c) is established in our lemma 5.1. Next, (d) was proved in [4] with reference to Lemma 4.2 in [4] for  $\eta = 0$ . But the latter is exactly *right-regularity* of  $g(\cdot)$ . Thus we completed the proof of the second implication in (i) as well as of the whole statement (i).

*Statement (ii).* It is enough to pass from  $A \in \mathcal{K}_\Sigma$  to  $A_* \in \mathcal{K}_\Sigma$  and apply already established statement (i).

*Statement (iii).* It stems immediately from the relation

$$(5.4) \quad \boxed{W^2(\lambda) \asymp w_*^2(\lambda), \lambda \in \mathbb{R}}$$

which is valid due to (3.11).

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